Fall 2015, MATH-566

Matroids I

Definition: Let E be a finite set and $\mathcal{F} \subseteq 2^E$. A set system $(E, \mathcal{F})$ is called a **matroid** if it satisfies

(M1) $\emptyset \in \mathcal{F}$;

(M2) If $X \subseteq Y \in \mathcal{F}$ then $X \in \mathcal{F}$;

(M3) If $X, Y \in \mathcal{F}$ and $|X| > |Y|$, then there is $x \in X \setminus Y$ with $Y \cup \{x\} \in \mathcal{F}$.

Elements in $\mathcal{F}$ are called **independent sets**.

1: Let E be any finite set and $k \geq 0$. Let $\mathcal{F} = \{F \subseteq E : |F| \leq k\}$. Show that $(E, \mathcal{F})$ is a matroid.
(It is called ***uniform matroid****.)*

2: Let E be the set of columns of a matrix A over some field.
Let $\mathcal{F} = \{F \subseteq E :$ the columns in F are linearly independent$\}$. Show that $(E, \mathcal{F})$ is a matroid.
(It is called ***linear*** *or* ***vector matroid****.)*

3: Let E be the set of edges of an undirected graph G. Let $\mathcal{F} = \{F \subseteq E : (V(G), F)$ is a forest$\}$. Show that $(E, \mathcal{F})$ is a matroid. *(It is called* ***graphic*** *or* ***cycle matroid****.)*

Maximal independent sets in $\mathcal{F}$ are called **bases** (see linear matroid).
Minimal dependent sets (means not independent) in $\mathcal{F}$ are called **circuits** or **cycles** (see graphic matroid).

Motivation for Matroids

Let $(E, \mathcal{F})$ be a matroid. Let $c : E \to \mathbb{R}_+$. Find $X \in \mathcal{F}$ such that $\sum_{e \in X} c(e)$ is maximized.
Notice that this would be the same as *maximum cost spanning tree* for graphic matroid.
$(E, \mathcal{F})$ being a matroid $\Rightarrow$ greedy algorithm works

1. Sort E such that $c(e_1) \geq c(e_2) \geq \cdots \geq c(e_m)$
2. Let $F = \emptyset$
3. for i in 1 to m
4. if $\{e_i\} \cup F \in \mathcal{F}$ then $F := F \cup \{e_i\}$.

4: Show that the greedy algorithm is correct. (Hint: similar to Kruskal's algorithm - consider optimal $F^\star$ and investigate the difference of F and $F^\star$.)

5: Let $(E, \mathcal{F})$ satisfy (M1) and (M2). Suppose that the greedy algorithm works for all $c : E \to \mathbb{R}_+$. Show that $(E, \mathcal{F})$ is a matroid.

6: Show that all bases of a matroid have the same cardinality.

Theorem 13.9 Let E be a finite set and $\mathcal{B} \subseteq 2^E$. $\mathcal{B}$ is the set of bases of some matroid iff $\mathcal{B}$ satisfies

(B1) $\mathcal{B} \neq \emptyset$;

(B2) For any $B_1, B_2 \in \mathcal{B}$ and any $x \in B_1 \setminus B_2$ there exists $y \in B_2$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

7: Show that matroids satisfy (B1) and (B2).

8: Show that if $\mathcal{B}$ satisfies (B1) and (B2), then all bases in $\mathcal{B}$ have the same cardinality.

9: For $\mathcal{B}$ satisfying (B1) and (B2) find a matroid $(E, \mathcal{F})$.